The general solution of the real $\mathbf{Z}_2^{\otimes N}$ graded contractions of $so(N+1)$

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Abstract

The general solution of the graded contraction equations for a $\mathbf{Z}_2^{\otimes N}$ grading of the real compact simple Lie algebra $so(N+1)$ is presented in an explicit way. It turns out to depend on 2^N-1 independent real parameters. The structure of the general graded contractions is displayed for the low dimensional cases, and kinematical algebras are shown to appear straightforwardly. The geometrical (or physical) meaning of the contraction parameters as curvatures is also analysed; in particular, for kinematical algebras these curvatures are directly linked to geometrical properties of possible homogeneous space-times.

1 Introduction

Graded contractions of (complex or real) Lie algebras have been introduced by de Montigny, Patera and Moody [1, 2] as a new approach encompassing the study of ordinary contractions of Lie algebras and allowing the contraction of representations to be simultaneously studied. The approach is based on the so-called *contraction equations*, which determine all possible *contracted* Lie algebras compatible with a given grading of some initial Lie algebra [3]. Ordinary (simple) Inönü-Wigner (IW) contractions [4] appear as related to a $\mathbf{Z}_2$ grading; the solution of the contraction equations is straightforward in this case. For more complicated grading groups, and in the complex case, these equations have been solved for several comparatively small grading groups (e.g. the complete list for $\mathbf{Z}_2$, $\mathbf{Z}_2 \otimes \mathbf{Z}_2$, $\mathbf{Z}_3$ is given in [1]), and a computer programme has been devised for handling more complicated cases [5].

In a previous paper the graded contractions of the *real* orthogonal algebra $so(N+1)$ associated to a fine grading group $\mathbf{Z}_2^{\otimes N}$ were studied without relying on a computer programme, and a *particular* set of solutions which depends on N real parameters was given [6]. To know the general solution of the contraction equations for this grading would be interesting as a first step to study similar graded contractions for algebras in the unitary $su(N+1)$ and symplectic series $sp(N+1)$, for which a natural $\mathbf{Z}_2^{\otimes N}$ grading can be "derived" from the orthogonal one [7]; the general solution of the orthogonal contraction equations could go a long way to provide a general solution for these other cases.

In this paper we advance the *general* solution for a fine $\mathbf{Z}_2^{\otimes N}$ grading of the real Lie algebra $so(N+1)$. It should be recalled that the general solution for a given graded algebra with grading group Γ is some subset of the corresponding list of solutions of the contraction equations for Γ, which generically depend only on the group Γ. Usually this is a *proper* subset; this is due to the possible presence of *irrelevant* contraction parameters which actually do not appear in the contractions equations for the given algebra (for instance, a given grading group element may have not a proper associated subspace, or two graded subspaces may commute in the initial Lie algebra). As far as we know, the list of generic solutions to the contraction equations for a $\mathbf{Z}_2^{\otimes N}$ grading group is not known. However, the strategy of solving generically the contraction equations can be succesfully bypassed in specific cases, as the example we are about to discuss clearly shows.

The paper is organized as follows. The structure of the $\mathbf{Z}_2^{\otimes N}$ grading of $so(N+1)$ and the corresponding contraction equations are presented in the next section. We solve the contraction equations in section 3 showing how the $3\binom{N+1}{3}$ initial relevant contraction parameters turn out to depend on $2^N - 1$ *independent* real parameters. All graded contractions are continuous in this case, and the number of possible contractions equals the expected number for a general composition of simple IW contractions; the result is in accordance with the not entirely obvious result in [8]: any continuous graded contraction is equivalent to some generalized IW contraction. The solution given in [6] appears as a rather particular case, as it corresponds to

having all but N parameters fixed to 1. Explicit results for the simplest cases with $N = 2, 3$ are given in section 4 in order to clearly describe the structure of the general solution; moreover for $N = 3$ we introduce all the (2+1) kinematical algebras [9] within the graded contracted algebras of $so(4)$ in a straightforward way, thus giving derivation of Lie algebra kinematics from the graded contraction perspective alternative to the one discussed in [10]. An interesting byproduct in this approach is the interpretation of the contraction coefficients as related to curvatures of homogeneous spaces.

2 The contraction equations

Recall that a *grading* of a real Lie algebra L by an Abelian finite group Γ [3] is a decomposition of the vector space structure of L:

$$L = \bigoplus_{\mu\in\Gamma} L_\mu, \tag{2.1}$$

such that if $x \in L_\mu$ and $y \in L_\nu$ then $[x, y]$ belongs to $L_{\mu+\nu}$:

$$[L_\mu, L_\nu] \subseteq L_{\mu+\nu}, \qquad \mu,\ \nu,\ \mu+\nu \in \Gamma. \tag{2.2}$$

A (real) *graded contraction* of the real Lie algebra L [1, 2] is a real Lie algebra L_ε with the same vector space structure as L, but with Lie brackets for $x \in L_\mu$ and $y \in L_\nu$ modified as follows:

$$[x, y]_\varepsilon := \varepsilon_{\mu,\nu}\, [x, y], \quad \text{in short hand form} \quad [L_\mu, L_\nu]_\varepsilon := \varepsilon_{\mu,\nu}\, [L_\mu, L_\nu], \tag{2.3}$$

where the *contraction parameters* $\varepsilon_{\mu,\nu}$ are real numbers such that L_ε is a Lie algebra; they must satisfy the *contraction equations*:

$$\varepsilon_{\mu,\nu} = \varepsilon_{\nu,\mu} \qquad\qquad \varepsilon_{\mu,\nu}\, \varepsilon_{\mu+\nu,\sigma} = \varepsilon_{\mu,\nu+\sigma}\, \varepsilon_{\nu,\sigma} \tag{2.4}$$

The Lie algebra $so(N+1)$ has $N(N+1)/2$ generators J_{ab} ($a, b = 0, 1, \ldots, N,\ a < b$) with non-zero Lie brackets:

$$[J_{ab}, J_{ac}] = J_{bc}, \qquad [J_{ab}, J_{bc}] = -J_{ac}, \qquad [J_{ac}, J_{bc}] = J_{ab}, \qquad a < b < c. \tag{2.5}$$

The fine grading group Γ of $so(N + 1)$ we are going to deal with is isomorphic to $\mathbf{Z}_2^{\otimes N}$ and is generated by a set of 2^N commuting involutive automorphisms $S_{\mathcal{S}} : so(N+1) \to so(N+1)$ where $\mathcal{S}$ is any subset of the set of indices $\mathcal{I} = \{0, 1, \ldots, N\}$ (see [6] for more details). The automorphism $S_{\mathcal{S}}$ is defined as

$$S_{\mathcal{S}} J_{ab} = (-1)^{\chi_{\mathcal{S}}(a)+\chi_{\mathcal{S}}(b)} J_{ab}, \tag{2.6}$$

where $\chi_{\mathcal{S}}(i)$ is the characteristic function over $\mathcal{S}$, which equals either 1 or 0 according as $i \in \mathcal{S}$ or $i \notin \mathcal{S}$.

Each involutive automorphism $S_{\mathcal{S}}$ provides a $\mathbf{Z}_2$ grading of $so(N+1) = L_0 \oplus L_1$ where L_0 is the $S_{\mathcal{S}}$-invariant subspace (spanned by the generators J_{ab} with either both indices or none in $S_{\mathcal{S}}$), while L_1 is the $S_{\mathcal{S}}$-anti-invariant subspace (spanned by the J_{ab} with a single index in $S_{\mathcal{S}}$). Note also that $S_{\mathcal{S}} \equiv S_{\mathcal{I}\backslash\mathcal{S}}$, i.e. the automorphism associated to a subset $\mathcal{S}$ is the same as the one associated to the complement $\mathcal{I}\backslash\mathcal{S}$ in the whole set of indices $\mathcal{I}$. For instance, if $\mathcal{I} = \{0,1,2,3\}$, then $S_{012} \equiv S_3$, $S_{13} \equiv S_{02}$, etc.

We choose the N automorphisms $S_0,\ S_{01},\ S_{012}, \ldots,\ S_{01\ldots N-1}$ as a basis for the Abelian grading group Γ of $so(N+1)$. Thus, a generic element $\mu \in \Gamma$ can be written as

$$\mu = \prod_{k=0}^{N-1} (S_{01\ldots k})^{\mu_k}, \qquad \mu_k \in \{0,1\}. \tag{2.7}$$

A generator J_{ab} of $so(N+1)$ belongs to the (one-dimensional) grading subspace L_μ where the sequence μ_k of "coordinates" of μ is characterized by a contiguous string of 1's starting at the ath position and ending at the $(b-1)$th position with 0's at the remaining places:

$$\langle J_{ab} \rangle = L_\mu \quad \equiv \quad \mu = \{0\ldots01_a\ldots10_b\ldots0\}. \tag{2.8}$$

Therefore we can denote each particular μ actually appearing in the decomposition (2.1) for the specific grading we are dealing with by the pair of indices $\mu \equiv ab,\ (a<b)$ instead of using its complete string. The contraction parameters $\varepsilon_{\mu,\nu}$ where at least one of μ, ν and $\mu+\nu$ is not of the form (2.8) are therefore irrelevant, as the corresponding grading subspace is the trivial null subspace. On the contrary, we call *relevant* contraction parameters those appearing in the contractions equations; they must have μ, ν and $\mu+\nu$ of the form (2.8). In [6] it was shown that in this case there are $3\binom{N+1}{3}$ *relevant* contraction parameters. They can be classified in three disjoint sets:

$$\alpha^a_{\ bc} \equiv \varepsilon_{ab,ac}, \qquad \beta_{ac}^{\ b} \equiv \varepsilon_{ab,bc}, \qquad \gamma_{ab}^{\ \ c} \equiv \varepsilon_{ac,bc}, \qquad a<b<c. \tag{2.9}$$

All contraction equations coming from (2.4) are naturally classed into groups of 12 equations, one group for each ordered set of *four* indices $a<b<c<d$:

$$\begin{array}{lll}
\beta_{ac}^{\ b}\beta_{ad}^{\ c} = \beta_{ad}^{\ b}\beta_{bd}^{\ c} & & \\
\alpha^a_{\ bc}\beta_{bd}^{\ c} = \alpha^a_{\ bd}\beta_{ad}^{\ c} & \alpha^a_{\ bd}\alpha^b_{\ cd} = \alpha^a_{\ cd}\beta_{ac}^{\ b} & \alpha^a_{\ bc}\alpha^b_{\ cd} = \alpha^a_{\ cd}\beta_{ad}^{\ b} \\
\alpha^a_{\ cd}\gamma_{bc}^{\ \ d} = \alpha^a_{\ bc}\gamma_{ab}^{\ \ d} & \alpha^a_{\ bd}\gamma_{bc}^{\ \ d} = \alpha^a_{\ bc}\gamma_{ac}^{\ \ d} & \\
\beta_{ad}^{\ b}\gamma_{ac}^{\ \ d} = \beta_{ac}^{\ b}\gamma_{bc}^{\ \ d} & \beta_{bd}^{\ c}\gamma_{ab}^{\ \ d} = \gamma_{ab}^{\ \ c}\gamma_{ac}^{\ \ d} & \beta_{ad}^{\ c}\gamma_{ab}^{\ \ d} = \gamma_{ab}^{\ \ c}\gamma_{bc}^{\ \ d} \\
\alpha^a_{\ cd}\beta_{bd}^{\ c} = \alpha^a_{\ bd}\gamma_{ab}^{\ \ c} & \alpha^b_{\ cd}\beta_{ad}^{\ c} = \beta_{ad}^{\ b}\gamma_{ab}^{\ \ c} & \alpha^b_{\ cd}\gamma_{ac}^{\ \ d} = \beta_{ac}^{\ b}\gamma_{ab}^{\ \ d}
\end{array} \tag{2.10}$$

In [6] the solution of these equations under the condition $\beta_{ac}^{\ b} \neq 0$ was derived; each such solution turns out to be equivalent to a solution with all $\beta_{ac}^{\ b} = 1$ and then the equations (2.10) dramatically simplify, so that all contraction parameters can be expressed in terms of N real independent parameters. However, when some $\beta_{ac}^{\ b}$ are allowed to be equal to zero, the equations are rather complicated, and the naive case-by-case analysis succesfully done for smaller grading groups is quickly realised an unfeasible.

3 The general solution

Consider real functions θ defined on the collection of all subsets of $\mathcal{I} = \{0, 1, \ldots, N\}$, $\theta : \mathcal{P}(\mathcal{I}) \longrightarrow \mathbb{R}$, satisfying the additional condition $\theta(\mathcal{S}) \equiv \theta(\mathcal{I} \backslash \mathcal{S})$. We denote the common value $\theta(\mathcal{S}) \equiv \theta(\mathcal{I} \backslash \mathcal{S})$ as $\theta_{\mathcal{S}}^{\mathcal{I} \backslash \mathcal{S}} \equiv \theta_{\mathcal{I} \backslash \mathcal{S}}^{\mathcal{S}}$; there are 2^N such values, one of which is $\theta_{\mathcal{I}}^{\emptyset}$. The general solution of the system (2.10) can be expressed in terms of these values, taken as independent parameters, according to following statement:

Theorem *The general solution of the $\mathbf{Z}_2^{\otimes N}$ graded contractions of the Lie algebra $so(N+1)$ depends on $2^N - 1$ real independent parameters $\theta_{\mathcal{S}}^{\mathcal{I} \backslash \mathcal{S}}$ where $\mathcal{I} = \{0, 1, \ldots, N\}$ and $\mathcal{S}$ is a proper subset of $\mathcal{I}$: $\mathcal{S} \subset \mathcal{I}$. The relevant contraction parameters are given by:*

$$
\begin{aligned}
\alpha^{a}_{\ bc} &= \prod_{\mathcal{S}} \theta_{\mathcal{S}}^{\mathcal{I} \backslash \mathcal{S}} = \prod \theta_{bc\ldots}^{a\ldots}, \quad \textit{with } \{b, c\} \subseteq \mathcal{S} \textit{ and } \{a\} \notin \mathcal{S}; \\
\beta_{ac}^{\ b} &= \prod_{\mathcal{S}} \theta_{\mathcal{S}}^{\mathcal{I} \backslash \mathcal{S}} = \prod \theta_{ac\ldots}^{b\ldots}, \quad \textit{with } \{a, c\} \subseteq \mathcal{S} \textit{ and } \{b\} \notin \mathcal{S}; \\
\gamma_{ab}^{\ \ c} &= \prod_{\mathcal{S}} \theta_{\mathcal{S}}^{\mathcal{I} \backslash \mathcal{S}} = \prod \theta_{ab\ldots}^{c\ldots}, \quad \textit{with } \{a, b\} \subseteq \mathcal{S} \textit{ and } \{c\} \notin \mathcal{S},
\end{aligned}
\tag{3.1}
$$

where the products with index $\mathcal{S}$ run over all possible (proper) subsets of $\mathcal{I}$ that satisfy the conditions imposed in each case. The non-identically zero Lie brackets of the contracted Lie algebra obtained from $so(N+1)$ are

$$
[J_{ab}, J_{ac}] = \alpha^{a}_{\ bc} J_{bc}, \quad [J_{ab}, J_{bc}] = -\beta_{ac}^{\ b} J_{ac}, \quad [J_{ac}, J_{bc}] = \gamma_{ab}^{\ \ c} J_{ab}, \quad a < b < c, \tag{3.2}
$$

without sum over repeated indices and with $\alpha^{a}_{\ bc}$, $\beta_{ac}^{\ b}$, $\gamma_{ab}^{\ \ c}$ given by (3.1).

Proof. The proof is rather direct but somewhat tedious; we restrict here to comment the main lines. Each of the contraction equations given in (2.10) is like $MN = PQ$, where each term carries three indices (two subindices and a third single superindex), taken out of four. The general solution of each such equation can be given in terms of eight parameters, $m^1, m_1, n^1, n_1, p^1, p_1, q^1, q_1$ as

$$
M = m^1 m_1, \qquad N = n^1 n_1, \qquad P = p^1 p_1, \qquad Q = q^1 q_1, \tag{3.3}
$$

which are however not independent, but must be subjected to four auxiliary relations

$$
m^1 = p^1, \qquad m_1 = q_1, \qquad n^1 = q^1, \qquad n_1 = p_1. \tag{3.4}
$$

Now repeat this decomposition for each equation (2.10), writing each parameter m, (resp. n, p, q) as a symbol ϑ with two groups of indices, the first one with the same index structure as M, (resp. N, P, Q) and taking for the second group the fourth index already present in the equation but not in M (resp. N, P, Q), placed either as a superindex or as a subindex, instead of the index 1 above. For instance, the first equation of (2.10) would lead to

$$
\beta_{ac}^{\ b} = \vartheta_{ac,}^{\ b,d} \vartheta_{ac,d}^{\ b,} \qquad \beta_{ad}^{\ c} = \vartheta_{ad,}^{\ c,b} \vartheta_{ad,b}^{\ c,} \qquad \beta_{ad}^{\ b} = \vartheta_{ad,}^{\ b,c} \vartheta_{ad,c}^{\ b,} \qquad \beta_{bd}^{\ c} = \vartheta_{bd,}^{\ c,a} \vartheta_{bd,a}^{\ c,} \tag{3.5}
$$

with the ϑ symbols satisfying

$$\vartheta_{ac,}^{\;b,d} = \vartheta_{ad,}^{\;b,c} \qquad \vartheta_{ac,d}^{\;b,} = \vartheta_{bd,a}^{\;c,} \qquad \vartheta_{ad,}^{\;c,b} = \vartheta_{bd,}^{\;c,a} \qquad \vartheta_{ad,b}^{\;c,} = \vartheta_{ad,c}^{\;b,} \tag{3.6}$$

As long as all auxiliary relations are satisfied, this transforms all contraction equations into identities, at the expense of introducing a rather large number of parameters, which are however subjected to a number of auxiliary relations (similar to (3.6)), which can be then eliminated in some adequate way. The result of the elimination boils down to two simple rules. First, each symbol $\vartheta_{ac,}^{\;b,d}$, $\vartheta_{ac,d}^{\;b,}$, ...actually depends on the two subsets of $\mathcal{I}$ made up with the union of all subindices and the union of all superindices, so that e.g. $\vartheta_{ab,c}^{\;d,} = \vartheta_{ac,b}^{\;d,} = \vartheta_{bc,a}^{\;d,}$ which will be simply denoted θ_{abc}^{d}, likewise, $\vartheta_{ac,}^{\;d,b} = \vartheta_{ac,}^{\;b,d}$ will be denoted θ_{ac}^{bd}. Second, each symbol θ_{ac}^{bd} depends only on the two subsets of indices, but not on their position as subindices or superindices, so that e.g. $\theta_{ac}^{bd} = \theta_{bd}^{ac}$.

However, these four index θ symbols are not independent. To see this, recall that each contraction equation involves four contraction coefficients. By using the previous device, each of these coefficients can be written as the product of two *four* index θ symbols. But each contraction coefficient appears several times in the whole set of contraction equations, and to each appearance a decomposition as a product of two θ symbols with four indices has been allocated. For instance, the coefficient β_{36}^{5} will appear in the first equation (2.10) for $abcd = 1356$ and also for $abcd = 3456$. For each of these appearances we have

$$\beta_{36}^{\;5} = \vartheta_{36,}^{\;5,1}\vartheta_{36,1}^{\;5,} = \theta_{36}^{15}\theta_{136}^{5} \qquad \beta_{36}^{\;5} = \vartheta_{36,}^{\;5,4}\vartheta_{36,4}^{\;5,} = \theta_{36}^{45}\theta_{346}^{5} \tag{3.7}$$

so the coefficients θ must satisfy

$$\theta_{36}^{15}\theta_{136}^{5} = \theta_{36}^{45}\theta_{346}^{5} \tag{3.8}$$

In the same way we will have to enforce the equality of many such products. This turns out in a number of quadratic equations, whose structure is again similar to the initial equations, but each involving *five* indices taken out of the set $\mathcal{I}$, so the same decomposition procedure can be applied again. For instance, each coefficient θ in the equation (3.8) would be decomposed as a product of two five index ϑ symbols, e.g., $\theta_{36}^{15} = \vartheta_{36,}^{15,4}\vartheta_{36,4}^{1,5}$, etc, where the set of five index coefficients ϑ must satisfy auxiliary equations, derived again from (3.4) and similar to (3.6). The elimination of these auxiliary equations boils down to the same simple rules stated before (e.g., $\vartheta_{36,}^{15,4} = \vartheta_{36,}^{14,5} = \ldots$, which will be denoted θ_{36}^{145}, etc. and $\theta_{36}^{145} = \theta_{145}^{36}$, etc.). Now all equations like (3.8) are turned into identities, and only the auxiliar equations for the five index θ symbols remain.

The process is iterated until no more indices are left, at which point all equations are transformed into identities; this explains the structure of the solution. It is easy to check that after using (3.1) all the contraction equations (2.10) are turned into identities.

It is worth remarking that there exists a close relationship between the parameters $\theta_{\mathcal{S}}^{\mathcal{I}\setminus\mathcal{S}}$ and the involutive automorphisms $S_{\mathcal{S}}$. Each non trivial involution $S_{\mathcal{S}}$ gives

rise to a simple IW contraction whose effect consists on a graded scale change with scaling factor λ on the anti-invariant generators under $S_{\mathcal{S}}$ (those J_{ab} where either a or b belongs to $\mathcal{S}$) followed by the limit $\lambda \to 0$. This scale change only modifies the parameter $\theta_{\mathcal{S}}^{\mathcal{I}\setminus\mathcal{S}} \to \lambda^2 \theta_{\mathcal{S}}^{\mathcal{I}\setminus\mathcal{S}}$, the remaining ones being invariant, and in the limit $\lambda \to 0$ this parameter vanishes. Thus, there are $2^N - 1$ simple IW contractions associated to the same number of non trivial involutions or of $\mathbf{Z}_2$ subgradings; the identity involution $S_{\mathcal{I}}$ would be associated with $\theta_{01\ldots N}^{\emptyset}$, which is the only value of $\theta_{\mathcal{S}}^{\mathcal{I}\setminus\mathcal{S}}$ not appearing explicitly in (3.1). The composition of two or more of such contractions is a generalized IW contraction [8] where more than one parameter $\theta_{\mathcal{S}}^{\mathcal{I}\setminus\mathcal{S}}$ go to zero at the same time (with possibly different powers of λ).

It is also clear that all graded contractions are continuous for the grading we are dealing with, as the identity element in the grading group has no an associated proper subspace.

The graded contractions of $so(N+1)$ with all $\theta_{\mathcal{S}}^{\mathcal{I}\setminus\mathcal{S}} \neq 0$ give rise to the different non-compact real forms $so(p,q)$ with $p+q = N+1$ (besides the original $so(N+1)$). In this case the graded contraction is not a contraction of the initial algebra in its original meaning of limiting process. When one or more $\theta_{\mathcal{S}}^{\mathcal{I}\setminus\mathcal{S}}$ are zero, a non simple Lie algebra is obtained. It is interesting to note that the whole family of graded contractions of $so(N+1)$ affords a sort of ordered "lattice" of algebras, starting at the simple real algebras $so(p,q)$ and ending at the extreme case, when all $\theta_{\mathcal{S}}^{\mathcal{I}\setminus\mathcal{S}} = 0$, into the Abelian algebra with all commutators zero.

If we only allow the N parameters $\theta_0^{12\ldots N}$, $\theta_{01}^{23\ldots N}$, $\theta_{012}^{34\ldots N}, \ldots, \theta_{012\ldots N-1}^{N}$ to take over arbitrary values enforcing the value 1 for all the remaining ones, the contracted algebra commutation relations are:

$$[J_{ab}, J_{ac}] = \kappa_{ab}\, J_{bc}, \quad [J_{ab}, J_{bc}] = -J_{ac}, \quad [J_{ac}, J_{bc}] = \kappa_{bc}\, J_{ab}, \quad a < b < c; \tag{3.9}$$

where the κ coefficients are defined as:

$$\kappa_{ab} := \kappa_{a+1}\kappa_{a+2}\ldots\kappa_b, \qquad a, b = 0, \ldots, N; \quad a < b, \tag{3.10}$$

$$\kappa_a := \theta_{0\ldots a-1}^{a\ldots N}, \qquad a = 1, \ldots, N. \tag{3.11}$$

In this way the so-called *Cayley–Klein algebras*, which were the particular case studied in [6], are recovered. As a collective, this subfamily of graded contractions inherits (in a more complicated form) most properties coming from the simple nature of the algebras $so(p,q)$, and is termed *quasisimple* in the literature [11].

4 Examples

Let us illustrate the results of the theorem for $so(3)$ and $so(4)$, where from the two ways of writing each coefficient θ, we have chosen the one where 0 appears as a subindex, even if this sometimes aparently spoils the simplicity of the rule (3.1).

4.1 $so(3)$

The grading group is $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ and is generated by the automorphisms S_0 and S_{01} acting on the generators $\{J_{01}, J_{02}, J_{12}\}$ as:

$$\begin{array}{l} S_0 : (J_{01}, J_{02}, J_{12}) \longrightarrow (-J_{01}, -J_{02}, J_{12}), \\ S_{01} : (J_{01}, J_{02}, J_{12}) \longrightarrow (J_{01}, -J_{02}, -J_{12}). \end{array} \tag{4.1}$$

These involutions endow the basis of $so(3)$ with the following grading:

$$L_{\{01\}} \equiv L_{12} = \langle J_{12} \rangle, \quad L_{\{10\}} \equiv L_{01} = \langle J_{01} \rangle, \quad L_{\{11\}} \equiv L_{02} = \langle J_{02} \rangle, \tag{4.2}$$

where the indices between brackets in L_μ denote the whole sequence $\{\mu_k\}$. There are $3\binom{2+1}{3} = 3$ relevant contraction coefficients (one α, one β and one γ) which depend on $2^2 - 1 = 3$ parameters $\theta_0^{12} \equiv \theta_{12}^0$, $\theta_{02}^1 \equiv \theta_1^{02}$ and $\theta_{01}^2 \equiv \theta_2^{01}$:

$$\begin{array}{l} \varepsilon_{\{10\},\{11\}} = \varepsilon_{01,02} \equiv \alpha^0{}_{12} = \theta_0^{12}, \\ \varepsilon_{\{10\},\{01\}} = \varepsilon_{01,12} \equiv \beta_{02}^{\,1} = \theta_{02}^1, \\ \varepsilon_{\{01\},\{11\}} = \varepsilon_{02,12} \equiv \gamma_{01}{}^2 = \theta_{01}^2. \end{array} \tag{4.3}$$

The commutation relations for the contracted algebra of $so(3)$ are:

$$[J_{01}, J_{02}] = \theta_0^{12} J_{12}, \quad [J_{01}, J_{12}] = -\theta_{02}^1 J_{02}, \quad [J_{02}, J_{12}] = \theta_{01}^2 J_{01}. \tag{4.4}$$

This family of algebras includes $so(3)$, $so(2,1)$, the Euclidean $e(2)$, Poincaré $p(1+1)$, Galilean $g(1+1)$ and the Abelian algebras. Upon graded contraction equivalence, the value θ_{02}^1 can be reduced either to 0 or 1, and then each of the two remaining contraction parameters θ_0^{12} and θ_{01}^2 can be reduced to either 1, 0 or -1.

This example allows to see clearly the point commented upon in the introduction: acording to the results in [1], for a generic $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ graded structure, there exists 40 non-equivalent solutions of the complex graded equations. Even if we are dealing with real graded contractions, the number of inequivalent solutions here is much lesser. The reason is easy to see: for the algebra $so(3)$ and the grading (4.2), the subspace $L_{\{00\}}$ is the trivial null subspace, so many contraction parameters which are relevant in the generic case, turn out to be irrelevant here. From another point of view, this case has been also discussed in [12].

It is worthy to analyze the meaning of the three contractions constants in this example; in fact the most interesting traits of the Nd case are already present in this simplest case. First, a notation change helps in the interpretation: we shall rewrite (4.4) as

$$[P_1, P_2] = \mu_1 J, \quad [P_1, J] = -\lambda P_2, \quad [P_2, J] = \mu_2 P_1. \tag{4.5}$$

Now for each algebra (4.5) we can build three two-dimensional symmetrical homogeneous spaces, each associated to the involutions S_0, S_{01}, S_{02}, taking the coset space of the graded contrated group by the subgroup generated by the elements

invariant under the involution. The first two spaces are called the space of points, and the space of lines. Each of these spaces has a canonical connection, as well as a compatible canonical (hierarchy of) metrics, coming from a suitably modified "Cartan-Killing" form, which is defined even for non-semisimple cases and reduces to the standard one for $so(3)$ and $so(2,1)$ [7]. Then the constants μ_1 and μ_2 turn out to be equal to the canonical curvature of the spaces of points and lines. The constant λ plays a similar role for the third homogeneous space corresponding to the involution S_{02} (the space of second kind lines).

4.2 $so(4)$

Let us consider now the $N=3$ case, with basis $\{J_{01}, J_{02}, J_{03}, J_{12}, J_{13}, J_{23}\}$. The group $\mathbf{Z}_2^{\otimes 3}$ determines the graded subspaces:

$$\begin{array}{lll} L_{\{100\}} \equiv L_{01} = \langle J_{01} \rangle, & L_{\{110\}} \equiv L_{02} = \langle J_{02} \rangle, & L_{\{111\}} \equiv L_{03} = \langle J_{03} \rangle, \\ L_{\{010\}} \equiv L_{12} = \langle J_{12} \rangle, & L_{\{011\}} \equiv L_{13} = \langle J_{13} \rangle, & L_{\{001\}} \equiv L_{23} = \langle J_{23} \rangle. \end{array} \tag{4.6}$$

There are $3\binom{3+1}{3} = 12$ relevant contraction parameters which can be written in terms of $2^3 - 1 = 7$ coefficients θ_0^{123}, θ_{01}^{23}, θ_{02}^{13}, θ_{03}^{12}, θ_{012}^{3}, θ_{013}^{2}, θ_{023}^{1} as follows:

$$\begin{array}{ll} \varepsilon_{\{100\},\{110\}} = \varepsilon_{01,02} \equiv \alpha^0{}_{12} = \theta_0^{123}\theta_{03}^{12} & \varepsilon_{\{100\},\{111\}} = \varepsilon_{01,03} \equiv \alpha^0{}_{13} = \theta_0^{123}\theta_{02}^{13} \\ \varepsilon_{\{110\},\{111\}} = \varepsilon_{02,03} \equiv \alpha^0{}_{23} = \theta_0^{123}\theta_{01}^{23} & \varepsilon_{\{010\},\{011\}} = \varepsilon_{12,13} \equiv \alpha^1{}_{23} = \theta_{01}^{23}\theta_{023}^{1} \\ \varepsilon_{\{010\},\{100\}} = \varepsilon_{01,12} \equiv \beta_{02}^{\ 1} = \theta_{02}^{13}\theta_{023}^{1} & \varepsilon_{\{011\},\{100\}} = \varepsilon_{01,13} \equiv \beta_{03}^{\ 1} = \theta_{03}^{12}\theta_{023}^{1} \\ \varepsilon_{\{001\},\{110\}} = \varepsilon_{02,23} \equiv \beta_{03}^{\ 2} = \theta_{03}^{12}\theta_{013}^{2} & \varepsilon_{\{001\},\{010\}} = \varepsilon_{12,23} \equiv \beta_{13}^{\ 2} = \theta_{02}^{13}\theta_{013}^{2} \\ \varepsilon_{\{110\},\{010\}} = \varepsilon_{02,12} \equiv \gamma_{01}{}^{2} = \theta_{01}^{23}\theta_{013}^{2} & \varepsilon_{\{111\},\{011\}} = \varepsilon_{03,13} \equiv \gamma_{01}{}^{3} = \theta_{01}^{23}\theta_{012}^{3} \\ \varepsilon_{\{111\},\{001\}} = \varepsilon_{03,23} \equiv \gamma_{02}{}^{3} = \theta_{02}^{13}\theta_{012}^{3} & \varepsilon_{\{011\},\{001\}} = \varepsilon_{13,23} \equiv \gamma_{12}{}^{3} = \theta_{03}^{12}\theta_{012}^{3} \end{array} \tag{4.7}$$

and for the contracted algebra we have the following non-identically vanishing Lie brackets:

$$\begin{array}{lll} [J_{01}, J_{02}] = \theta_0^{123}\theta_{03}^{12} J_{12} & [J_{01}, J_{12}] = -\theta_{02}^{13}\theta_{023}^{1} J_{02} & [J_{02}, J_{12}] = \theta_{01}^{23}\theta_{013}^{2} J_{01} \\ [J_{01}, J_{03}] = \theta_0^{123}\theta_{02}^{13} J_{13} & [J_{01}, J_{13}] = -\theta_{03}^{12}\theta_{023}^{1} J_{03} & [J_{03}, J_{13}] = \theta_{01}^{23}\theta_{012}^{3} J_{01} \\ [J_{02}, J_{03}] = \theta_0^{123}\theta_{01}^{23} J_{23} & [J_{02}, J_{23}] = -\theta_{03}^{12}\theta_{013}^{2} J_{03} & [J_{03}, J_{23}] = \theta_{02}^{13}\theta_{012}^{3} J_{02} \\ [J_{12}, J_{13}] = \theta_{01}^{23}\theta_{023}^{1} J_{23} & [J_{12}, J_{23}] = -\theta_{02}^{13}\theta_{013}^{2} J_{13} & [J_{13}, J_{23}] = \theta_{03}^{12}\theta_{012}^{3} J_{12} \end{array} \tag{4.8}$$

For each three dimensional subalgebra, a structure like (4.4) is found, the structure constants being now products of two parameters θ.

It is interesting to find out how the eleven (2+1) kinematical algebras [9] appear as particular graded contracted algebras of $so(4)$. Let us take a physical basis $\{H, P_1, P_2, K_1, K_2, J_3\}$ whose elements generate the time translation, two space translations, two boosts and one space rotation, respectively. We can consider the following identification with the "abstract" basis $\{J_{ab}\}$:

$$H \equiv J_{01}, \quad P_1 \equiv J_{02}, \quad P_2 \equiv J_{03}, \quad K_1 \equiv J_{12}, \quad K_2 \equiv J_{13}, \quad J_3 \equiv J_{23}. \tag{4.9}$$

The most restrictive requirement to be imposed on these algebras is the automorphism condition on parity and time reversal; this is automatically taken into account using graded contractions, and these transformations correspond to the automorphisms S_{01} and S_{023} respectively. Ordinary physical space isotropy is translated into the requirement

$$[J_3, H] = 0, \qquad [J_3, P_i] = \epsilon_{3ij} P_j, \qquad [J_3, K_i] = \epsilon_{3ij} K_j. \tag{4.10}$$

This condition implies

$$\theta_{03}^{12}\theta_{013}^{2} = \theta_{02}^{13}\theta_{012}^{3} = \theta_{02}^{13}\theta_{013}^{2} = \theta_{03}^{12}\theta_{012}^{3} = 1. \tag{4.11}$$

Then the four coefficients θ_{02}^{13}, θ_{03}^{12}, θ_{012}^{3}, θ_{013}^{2} must be simultaneously different from zero, and are determined by any of them, which by means of a simple rescaling can be made equal to 1; we will asume therefore that all these contraction coefficients rest equal to 1 (this way euclidean-like space isotropy will be preserved under graded contractions).

So any possible kinematical algebra is completely described by the values of three *independent* contraction constants, θ_0^{123}, θ_{01}^{23} and θ_{023}^{1}, a rather expected outcome as imposing space isotropy we are indeed reducing the problem to a (1+1) kinematics, where the most general solution (4.4) depends on three contraction parameters. The commutators (4.8) in the basis (4.9) read:

$$\begin{array}{lll} [H, P_i] = \theta_0^{123}\, K_i, & [H, K_i] = -\theta_{023}^{1}\, P_i, & [P_i, K_j] = \delta_{ij}\theta_{01}^{23}\, H, \\ [P_1, P_2] = \theta_0^{123}\theta_{01}^{23}\, J_3, & [K_1, K_2] = \theta_{01}^{23}\theta_{023}^{1}\, J_3, & i,j = 1,2. \end{array} \tag{4.12}$$

Finally the boosts generate a non-compact group when $\theta_{01}^{23}\theta_{023}^{1} \leq 0$, so this condition should be also enforced.

A more clear view is obtained by performing now a notational change: $\theta_0^{123} \equiv \mu_1$, $\theta_{01}^{23} \equiv \mu_2$ and $\theta_{023}^{1} \equiv \lambda$, so that the (1+1) kinematical subalgebra generated by H, P_1, K_1 (and also H, P_2, K_2) closes a Lie algebra (4.5):

$$\begin{array}{lll} [H, P_i] = \mu_1\, K_i, & [H, K_i] = -\lambda\, P_i, & [P_i, K_j] = \delta_{ij}\mu_2\, H, \\ [P_1, P_2] = \mu_1\mu_2\, J_3, & [K_1, K_2] = \mu_2\lambda\, J_3, & i,j = 1,2. \end{array} \tag{4.13}$$

The constant λ can be reduced either to zero or to 1 by means of an equivalence; we will always asume it takes on these values. The physical meaning of contraction parameters can be also clearly seen in the commutation relations (4.13) as linked to geometrical properties of the corresponding homogeneous spaces. The most important is space-time itself: space-time curvature (i.e., along 2-flat directions like (tx) and (ty)) equals to μ_1, so De Sitter and Newton-Hooke universes have non-zero constant space-time curvature, while Galilei or Minkowski have zero curvature. Two-dimensional space curvature (along the 2-flat (xy), we could say space-space curvature) equals the product $\mu_1\mu_2$, and can therefore be zero when $\mu_2 = 0$, even if the space-time curvature is different from zero; this is the case in the non-relativistic Newton-Hooke algebras, which has a flat 2-space. However, when $\mu_2 \neq 0$, space-time curvature and space curvature are linked, and appear simultaneously.

Likewise, the constant μ_2 is the curvature of the space of lines, which is negative in the "relativistic" space-times and zero in the non-relativistic ones; positive values are not allowed as they would lead to compact inertial transformations. In fact the curvature of the space of lines is linked to the fundamental constant c of relativistic theories as $\mu_2 = \frac{-1}{c^2}$, and special relativity is no more than stating that the kinematical space of lines has a constant non-zero (negative) curvature.

Each of the possible values of the pair μ_2, λ is coupled with the three possible values for the ordinary space-time curvature μ_1. This way, all graded contraction constants appear as physical parameters, whose values are determined by the geometrical properties of space-time itself.

We summarize the results in table I, where the usual physical name of the algebra is displayed along with the values of the contraction coefficients. The first six algebras are "relative-time" type, while the six remaining are "absolute-time" type. The space-time, speed-space and speed-time contractions correspond, in this order, to the cancelation of μ_1, μ_2 and λ, and the algebras are classed naturally in four groups of three algebras, corresponding to the three essentially different values of μ_1; The unphysical para-Galileicase is somewhat exceptional as both $\mu_1 = 1$ and $\mu_1 = -1$ lead to isomorphic algebras.

Table I. Kinematical algebras as graded contractions of $so(4)$.

Kinematical algebra		μ_1	μ_2	λ
De Sitter	$so(3,1)$	-1	-1	1
Poincaré	$iso(2,1)$	0	-1	1
Anti-De Sitter	$so(2,2)$	1	-1	1
Inhomogeneous $so(3)$	$iso(3)$	-1	-1	0
Carroll	$ii'so(2)$	0	-1	0
Para-Poincaré	$iso(2,1)$	1	-1	0
Expanding Newton–Hooke	$t_4(so(2) \oplus so(1,1))$	-1	0	1
Galilei	$iiso(2)$	0	0	1
Oscillating Newton–Hooke	$t_4(so(2) \oplus so(2))$	1	0	1
Para-Galilei	$iiso(2)$	-1	0	0
Static		0	0	0
Para-Galilei	$iiso(2)$	1	0	0

The kinematical algebras in higher dimensions can be obtained in a similar way. We would also like to recall that a family of $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ graded contractions of any real form of the complex Lie algebra B_2 contains the (3+1) kinematical algebras; this procedure was used in [10].

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